LEZIONE 30

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ESERCIZIO:

1) Calcolare $\int \frac{1}{x^2-2x}\,dx$

2) Calcolare $\int \frac{1}{e^{2x}-2}\,dx$

$$\int \frac{1}{x^2-2x}\,dx \qquad x^2-2x = x(x-2)$$

$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$= \frac{A(x-2)+Bx}{x(x-2)} = \frac{(A+B)x-2A}{x(x-2)}$$

$$\begin{cases} A+B=0 \\ -2A=1 \end{cases} \Rightarrow \begin{cases} B=-A=\frac{1}{2} \\ A=-\frac{1}{2} \end{cases}$$

$$\int \frac{1}{x^2-2x}\,dx = \int -\frac{1}{2}\frac{1}{x} + \frac{1}{2}\frac{1}{x-2}\,dx$$

$$= -\frac{1}{2}\log|x| + \frac{1}{2}\log|x-2| + C$$

$$\int \frac{1}{e^{2x}-2}\,dx = \qquad y=e^{2x}$$

$$dy = e^{2x}\cdot 2\,dx$$

$$dx = \frac{1}{2e^{2x}}\,dy = \frac{1}{2y}\,dy$$

$$= \int \frac{1}{y-2}\,\frac{1}{2y}\,dy$$

$$= \frac{1}{2}\int \frac{1}{y(y-2)}\,dy = \frac{1}{2}\left[-\frac{1}{2}\log|y| + \frac{1}{2}\log|y-2| + C\right]$$

$$= -\frac{1}{4}\log|y| + \frac{1}{4}\log|y-2| + C$$

$$= -\frac{1}{4}\log(e^{2x}) + \frac{1}{4}\log|e^{2x}-2| + C$$

$$= -\frac{1}{2}x + \frac{1}{4}\log|e^{2x}-2| + C.$$

ESERCIZIO

Calcolare $\displaystyle\int \underbrace{x}_{f'} \log(2+3x)\,dx$

$$= \frac{1}{2}x^2 \log(2+3x) - \int \frac{1}{2}x^2 \frac{1}{2+3x}\cdot 3\,dx$$

$$= \frac{1}{2}x^2 \log(2+3x) - \frac{3}{2}\underbrace{\int \frac{x^2}{2+3x}\,dx}_{I}$$

$$\begin{array}{rrr|l}
x^2 & +\,0x & +\,0 & 3x+2 \\
\hline
x^2 & +\,\frac{2}{3}x & & \frac{x}{3}-\frac{2}{9} \\
\hline
// & -\frac{2}{3}x & +\,0 & \\
 & -\frac{2}{3}x & -\,\frac{4}{9} & \\
\hline
 & // & +\,\frac{4}{9} &
\end{array}$$

$$x^2 = (3x+2)\left(\frac{x}{3}-\frac{2}{9}\right) + \frac{4}{9}$$

$$\frac{x^2}{3x+2} = \frac{x}{3} - \frac{2}{9} + \frac{\frac{4}{9}}{3x+2}$$

$$I = \int \frac{x^2}{3x+2}\,dx = \frac{1}{6}x^2 - \frac{2}{9}x + \frac{4}{9}\int \frac{1}{3x+2}\,dx$$

$$= \frac{1}{6}x^2 - \frac{2}{9}x + \frac{4}{27}\log|3x+2| + C$$

$$\int x \log(2+3x)\,dx = \frac{1}{2}x^2\log(2+3x) - \frac{1}{4}x^2 + \frac{1}{3}x - \frac{2}{9}\log|3x+2| + C$$

ESERCIZIO:

Calcolare $\displaystyle\int \frac{1}{2e^x + e^{-x} + 1}\,dx$

$$= \int \frac{1}{2e^x + \frac{1}{e^x} + 1}\, dx = \int \frac{1}{\frac{2e^{2x}+1+e^x}{e^x}}\, dx$$

$$= \int \frac{\overbrace{e^x \, dx}^{dt}}{2e^{2x}+e^x+1} \qquad \begin{aligned} t &= e^x \\ dt &= e^x\, dx \end{aligned}$$

$$= \int \frac{1}{2t^2+t+1}\, dt \qquad 2t^2+t+1 \qquad \Delta = 1 - 4\cdot 2 = -7 < 0$$

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

$$\begin{aligned} 2t^2+t+1 &= 2\left(\left(t+\frac{1}{4}\right)^2 + \frac{7}{16}\right) \\ &= 2\cdot\frac{7}{16}\left(\frac{16}{7}\left(t+\frac{1}{4}\right)^2 + 1\right) \\ &= \frac{7}{8}\left(\left(\frac{4t+1}{\sqrt{7}}\right)^2 + 1\right) \end{aligned}$$

$$\begin{aligned} \int \frac{1}{2t^2+t+1}\, dt &= \frac{8}{7}\int \frac{1}{\left(\frac{4t+1}{\sqrt{7}}\right)^2+1}\, dt \\ &= \frac{8}{7}\arctan\left(\frac{4t+1}{\sqrt{7}}\right) \Big/ \frac{4}{\sqrt{7}} + C \\ &= \frac{8}{7}\cdot\frac{\sqrt{7}}{4}\arctan\left(\frac{4t+1}{\sqrt{7}}\right) + C \\ &= \frac{2}{\sqrt{7}}\arctan\left(\frac{4e^x+1}{\sqrt{7}}\right) + C \end{aligned}$$

ESERCIZIO

Calcolare $\displaystyle\int_1^e \overbrace{\frac{1}{x}\,\frac{\log x - 1}{\log^2 x - 2\log x + 10}\, dx}^{dy}$ $\qquad \begin{aligned} y &= \log x \\ dy &= \frac{1}{x}\, dx \end{aligned}$

$$= \int_0^1 \frac{y-1}{y^2-2y+10}\, dy \qquad \Delta = 4 - 40 < 0.$$

$$= \frac{1}{2}\int_0^1 \frac{2y-2}{y^2-2y+10}\,dy = \frac{1}{2}\log|y^2-2y+10|\Big|_0^1$$

$$= \frac{1}{2}\log 9 - \frac{1}{2}\log 10 = \frac{1}{2}\log\frac{9}{10}$$

Come calcolare $\int \frac{p(x)}{q(x)}\,dx$ se $\deg(q(x)) \geq 3$.

1) A meno di fare la divisione tra p e q possiamo sempre ricondurci al caso in cui $\deg(p(x)) < \deg(q(x))$.

2) Scomporre $q(x)$:

$$q(x) = a\,(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}\cdots(x-x_k)^{\alpha_k}\,q_1(x)^{\beta_1}\,q_2(x)^{\beta_2}\cdots q_s(x)^{\beta_s}$$

dove $q_1(x), \dots, q_s(x)$ sono polinomi di grado 2 irriducibili (cioè $\Delta < 0$).

Idea: Possiamo scrivere $\frac{p(x)}{q(x)}$ come

$$\frac{p(x)}{q(x)} = \frac{A_{11}}{x-x_1} + \frac{A_{12}}{(x-x_1)^2} + \frac{A_{13}}{(x-x_1)^3} + \cdots + \frac{A_{1,\alpha_1}}{(x-x_1)^{\alpha_1}}$$

$$+ \frac{A_{21}}{x-x_2} + \frac{A_{22}}{(x-x_2)^2} + \cdots + \frac{A_{2,\alpha_2}}{(x-x_2)^{\alpha_2}}$$

$$\vdots$$

$$+ \frac{A_{k1}}{x-x_k} + \frac{A_{k,2}}{(x-x_k)^2} + \cdots + \frac{A_{k,\alpha_k}}{(x-x_k)^{\alpha_k}}$$

$$+ \frac{B_{11}q_1'(x) + C_{11}}{q_1(x)} + \frac{B_{12}q_1'(x) + C_{12}}{q_1(x)^2} + \cdots + \frac{B_{1,\beta_1}q_1'(x) + C_{1,\beta_1}}{q_1(x)^{\beta_1}}$$

$$+ \cdots$$

$$\vdots$$

$$+ \frac{B_{s,1}q_s'(x)+C_{s,1}}{q_s(x)} + \frac{B_{s,2}q_s'(x)+C_{s,2}}{q_2(x)^2} + \cdots + \frac{B_{s,\beta_s}q_s'(x)+C_{s,\beta_s}}{q_s(x)^{\beta_s}}$$

ESEMPIO

$$\int \frac{1}{(2x+1)(x^2+x+1)}\,dx$$

$$\frac{1}{(2x+1)(x^2+x+1)} = \frac{A}{2x+1} + \frac{B(2x+1)+C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1) + B(2x+1)^2 + C(2x+1)}{(2x+1)(x^2+x+1)}$$

$$= \frac{Ax^2+Ax+A + B(4x^2+1+4x) + 2Cx+C}{(2x+1)(x^2+x+1)}$$

$$= \frac{(A+4B)x^2 + (A+4B+2C)x + A+B+C}{(2x+1)(x^2+x+1)}$$

$$\begin{cases} A+4B=0 \\ A+4B+2C=0 \\ A+B+C=1 \end{cases} \Longleftrightarrow \begin{cases} A=-4B \\ C=0 \\ A+B=1 \end{cases} \Longrightarrow \begin{cases} A=-4B \\ C=0 \\ -3B=1 \end{cases}$$

$$\Rightarrow \begin{cases} A=\frac{4}{3} \\ C=0 \\ B=-\frac{1}{3} \end{cases}$$

$$\int \frac{1}{(2x+1)(x^2+x+1)}\,dx = \int \frac{\frac{4}{3}}{2x+1} - \frac{\frac{1}{3}(2x+1)}{x^2+x+1}\,dx$$

$$= \frac{2}{3}\log|2x+1| - \frac{1}{3}\log|x^2+x+1| + C$$

$$= \frac{2}{3}\log|2x+1| - \frac{1}{3}\log(x^2+x+1) + C.$$

ESEMPIO

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx$$

$$\frac{6x}{(x-2)^2(x^2+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{2Cx+D}{x^2+2}$$

Facendo i conti si trova che $\begin{cases} A = -\frac{1}{3} \\ B = 2 \\ C = \frac{1}{6} \\ D = -\frac{4}{3} \end{cases}$

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx = \int -\frac{1}{3}\frac{1}{x-2} + \frac{2}{(x-2)^2} + \frac{1}{6}\frac{2x}{x^2+2} - \frac{4}{3}\frac{1}{x^2+2}\,dx$$
$$= -\frac{1}{3}\log|x-2| - \frac{2}{(x-2)} + \frac{1}{6}\log(x^2+2) - \frac{4}{3}\int\frac{1}{x^2+2}\,dx$$

$$\boxed{\int y^{-2}dy = \frac{y^{-2+1}}{-2+1} = -y^{-1} = -\frac{1}{y}}$$

$$\frac{1}{2}\Big/\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\int\frac{1}{x^2+2}\,dx = \frac{1}{2}\int\frac{1}{\frac{x^2}{2}+1}\,dx$$
$$= \frac{1}{2}\int\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx$$
$$= \frac{1}{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\Big/\frac{1}{\sqrt{2}} + c$$
$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c.$$

Quindi:

$$\int\frac{6x}{(x-2)^2(x^2+2)}\,dx = -\frac{1}{3}\log|x-2| - \frac{2}{x-2} + \frac{1}{6}\log(x^2+2) - \frac{4}{3\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.$$

Alcune sostituzioni standard.

1) $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ può essere comodo fare la sostituzione $y = \sqrt[n]{\frac{ax+b}{cx+d}}$.

ESEMPIO

$$\int\frac{1}{2+3\sqrt{x}}\,dx$$
$$= \int\frac{1}{2+3y}\,2y\,dy$$

$y = \sqrt{x}$ (oppure $x = y^2$, $dx = 2y\,dy$)

$dy = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,dy = 2y\,dy$

$$= 2\int \frac{y}{2+3y}\,dy = \frac{2}{3}\int \frac{3y}{2+3y}\,dy$$

$$= \frac{2}{3}\int \frac{3y+2-2}{2+3y}\,dy$$

$$= \frac{2}{3}\int \left(1 - \frac{2}{2+3y}\right)dy$$

$$= \frac{2}{3}\left(y - \frac{2}{3}\log|2+3y|\right) + C$$

$$= \frac{2}{3}y - \frac{4}{9}\log|2+3y| + C$$

$$= \frac{2}{3}\sqrt{x} - \frac{4}{9}\log|2+3\sqrt{x}| + C$$

$$= \frac{2}{3}\sqrt{x} - \frac{4}{9}\log(2+3\sqrt{x}) + C$$

ESEMPIO

$$\int \frac{1}{(x+6)\sqrt{x+2}}\,dx$$

$$y = \sqrt{x+2}$$

$$dy = \frac{1}{2\sqrt{x+2}}\cdot 1\,dx = \frac{1}{2\sqrt{x+2}}\,dx$$

$$\frac{1}{\sqrt{x+2}}\,dx = 2\,dy$$

$$y = \sqrt{x+2} \Rightarrow y^2 = x+2$$

$$x = y^2 - 2$$

$$= \int \frac{1}{(y^2-2+6)}\,2\,dy$$

$$= 2\int \frac{1}{y^2+4}\,dy$$

$$= \frac{1}{2}\int \frac{1}{\frac{y^2}{4}+1}\,dy = \frac{1}{2}\int \frac{1}{\left(\frac{y}{2}\right)^2+1}\,dy = \frac{1}{2}\arctan\left(\frac{y}{2}\right)\cdot 2 + C$$

$$= \arctan\left(\frac{y}{2}\right) + C$$

$$= \arctan\left(\frac{\sqrt{x+2}}{2}\right) + C.$$

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2) Formule parametriche per $\sin x$ e $\cos x$ in termini di:

$\tan\frac{x}{2}$.

Per integrali del tipo $\int R(\sin x, \cos x)\, dx$ può essere utile la sostituzione $t = \tan\frac{x}{2}$ ($x = 2\arctan t$). Si usano le formule:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}\, dt.$$

ESEMPIO

$$\int \frac{1}{\sin x}\, dx \qquad t = \tan\frac{x}{2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt = \int \frac{\cancel{1+t^2}}{\cancel{2}t} \cdot \frac{\cancel{2}}{\cancel{1+t^2}}\, dt$$

$$= \int \frac{1}{t}\, dt = \log|t| + C = \log\left|\tan\frac{x}{2}\right| + C.$$

ESEMPIO

$$\int \frac{1}{\cos x}\, dx = \qquad t = \tan\frac{x}{2}$$

$$\cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1}{\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{2}{1-t^2}\, dt. \qquad 1-t^2 = (1-t)(1+t)$$

$$\frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t}$$

$$= \frac{A(1+t) + B(1-t)}{1-t^2} = \frac{(A-B)t + A+B}{1-t^2}$$

$$\begin{cases} A-B=0 \\ A+B=1 \end{cases} \Rightarrow \begin{cases} B = \frac{1}{2} \\ A = \frac{1}{2} \end{cases}$$

$$\int \frac{1}{1-t^2}\,dt = \int \frac{1}{2}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t}\,dt$$

$$\int \frac{2}{1-t^2}\,dt = \int \frac{1}{1-t} + \frac{1}{1+t}\,dt$$

$$= -\log|1-t| + \log|1+t| + C$$

$$\int \frac{1}{\cos x}\,dx = \int \frac{2}{1-t^2}\,dt = -\log|1-t| + \log|1+t| + C$$

$$= -\log\left|1-\tan\frac{x}{2}\right| + \log\left|1+\tan\frac{x}{2}\right| + C.$$

$$\int \frac{1}{2\cos x + 5}\,dx \qquad t = \tan\frac{x}{2} \qquad dx = \frac{2}{1+t^2}\,dt \qquad \left(\begin{aligned} dt &= \left(1+\tan^2\frac{x}{2}\right)\frac{1}{2}\,dx \\ dx &= \frac{2}{1+\tan^2\frac{x}{2}}\,dt \\ &= \frac{2}{1+t^2}\,dt \end{aligned}\right)$$

$$= \int \frac{1}{2\cdot\frac{1-t^2}{1+t^2} + 5} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1+t^2}{2(1-t^2) + 5(1+t^2)} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{2-2t^2+5+5t^2}\,dt = \int \frac{2}{3t^2+7}\,dt.$$

$$= \frac{2}{7}\int \frac{1}{\frac{3}{7}t^2+1}\,dt = \frac{2}{7}\int \frac{1}{\left(\sqrt{\frac{3}{7}}t\right)^2+1}\,dt$$

$$= \frac{2}{7}\arctan\left(\sqrt{\frac{3}{7}}t\right)/\sqrt{\frac{3}{7}} + C$$

$$= \frac{2}{7}\cdot\sqrt{\frac{7}{3}}\arctan\left(\sqrt{\frac{3}{7}}t\right) + C$$

$$= \frac{2}{\sqrt{21}}\arctan\left(\sqrt{\frac{3}{7}}\tan\frac{x}{2}\right) + C.$$

Altra sostituzione:

$$\int R\left(x, \sqrt{a^2 - x^2}\right)$$ utile la sostituzione $t = a \sin x$.

Non sempre queste sostituzioni sono le migliori.

$$\int \frac{\cos x}{1 + 4 \sin x} \, dx$$ La sostituzione migliore è $t = \sin x$
$dt = \cos x \, dx$

$$= \int \frac{1}{1+4t} \, dt = \frac{1}{4} \log |1+4t| + C$$

$$= \frac{1}{4} \log(1 + 4 \sin x) + C$$

Con la sostituzione $t = \tan \frac{x}{2}$ in questo caso l'integrale è più complicato da risolvere.